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Magnetohydrodynamic flow in rectangular ducts. II

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This paper is an extension of an earlier paper by Hunt (1965) on laminar motion of a conducting liquid in a rectangular duct under a uniform transverse magnetic field. The effects of the duct having conducting walls are further explored; in this case the duct considered has perfectly conducting walls parallel to the field and non-conducting walls perpendicular to the field. A solution is obtained for high Hartmann numbers by analysing the boundary layers on the walls. This solution involves an integral equation of a standard form.

It is found that in this case, unlike the cases studied in the earlier paper, the velocity profiles in the boundary layers are monotonically decreasing. The effect of an external electrical circuit is examined, although it is found that it does not influence the form of the velocity profiles.

1. Introduction

The fully-developed laminar flow of uniformly conducting and incompressible fluids through ducts under the action of a transverse magnetic field is attracting considerable interest at the present time, mainly for two reasons.

First, magnetohydrodynamic generators, pumps and accelerators are devices of practical importance in which conducting fluids are passed through transverse magnetic fields. The analysis of the flow in these devices is formidable for one may have to take into account the variable conductivity and density of the fluid, complicated potential drops between the electrodes and the fluid and the fact that the flow is usually turbulent. In order to make progress in the understanding of the phenomena therefore, considerable simplification is necessary which may take various forms, e.g. (*a*) an assumption of slug flow (Neuringer & Migotsky 1963), (*b*) a reduction of the problem to one-dimensional gas dynamics (Resler & Sears 1958), (*c*) a two-dimensional analysis of the development of the laminar boundary layer on the walls in the direction of the flow (Kerrebrock 1961; Hale & Kerrebrock 1964), (*d*) a two-dimensional analysis of the flow down the duct assuming that it is inviscid (Sutton & Carlson 1961), or (*e*) the form used in the present paper, two-dimensional analysis of the flow variation across the duct assuming that it is laminar, fully developed and that there is no variation of fluid properties throughout the duct. These various idealizations are comple-

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mentary in that they each extract some of the basic physical ideas, and collectively it is hoped that they will prove useful in interpreting more complicated physical situations.

Secondly, this theory of duct flows can be tested in laboratory experiments with liquid metals. The uncertainty in the experimental results can be reduced to below 1 %, and consequently these experiments provide critical tests for the theory, in marked contrast with the majority of magneto-fluid dynamic experiments.

Since most magnetohydrodynamic generators and pumps have a rectangular cross-section, we shall confine ourselves to examining rectangular ducts. Exact solutions have been obtained for laminar flows of uniformly conducting incompressible fluids through rectangular ducts with thin conducting walls under transverse magnetic fields by Chang & Lundgren (1961), Uflyand (1961) and Hunt (1965). Chang & Lundgren and Uflyand analysed the case in which all the walls were perfectly conducting. Hunt analysed (i) the case in which the walls perpendicular to the field (walls *BB*, in figure 1) were perfectly conducting and those parallel to the magnetic field (walls *AA*) were thin and of arbitrary conductivity, and (ii) the case in which the walls *BB* were thin and of arbitrary conductivity and the walls *AA* were non-conducting. Thus he included the previous author's analysis as a special case of (i). Hunt also examined the form of the solutions for large M, where M is the Hartmann number, and found that varying the conductivity of the walls *AA* dramatically altered the form of the velocity profile in the boundary layers on walls *AA* and also the velocity flux through them. When the walls *AA* are non-conducting and the walls *BB* perfectly conducting, he found that large positive and negative velocities of order Mv_c are induced, where v_c is the velocity of the core. This fact indicates that the magnetic field may destabilize the flow in certain types of duct. It is this effect of the conductivity of the walls on the flow which gives the problem its physical interest and suggests the need for solving the outstanding problems.

In ducts of most practical value the walls *AA* are conducting and the walls *BB* are non-conducting; this case is not included in any of those examined by Hunt and at present no complete analytic solution is available. Grinberg (1961, 1962) has, however, reduced the problem to the solution of an integral equation, whose kernel is the Green's function for the problem and involves a double infinite series of Bessel funtions. When the Hartmann number M is large, only the leading terms of this series need be retained and he was then able to solve the simpler equation. In order to determine the current and velocity distribution and the mass flux down the tubes, however, further numerical work needs to be done. In this paper we approach the problem of the flow at high Hartmann numbers using a boundary-layer technique which has the advantage that the analysis is more transparent and it is easier to form a physical picture of the properties of the magnetic and velocity fields. Expressions are obtained for the leading terms in the expansion of the flux through the duct in descending powers of $M^{\frac{1}{2}}$. Also diagrams and a graph are displayed showing representative velocity and magnetic fields in the neighbourhood of the walls *AA* where their structure is complicated.

Usually the walls *AA* are electrically connected and either the duct supplies current to a load or a potential difference is placed across the walls *AA* to drive the flow. We show that if the walls *AA* are sufficiently highly conducting, the external electric circuit has no effect on the mathematical problem and that it is a trivial calculation to work out its effect on the flow parameters. Some examples of external circuits are given. In comparing the cases where the walls *AA* are conducting and the walls *BB* are non-conducting and where all the walls are conducting we find as before, that the conductivity of the walls has a marked effect on the flow in the boundary layers on the walls *AA*; we also find that in some cases the conductivity of the walls *in the corners* is important since the current distribution in the corners affects the rest of the flow in the boundary layer.

2. The formulation of the problem and the basic solution

We consider the steady flow of an incompressible conducting fluid driven by a pressure gradient along a rectangular duct under an imposed transverse magnetic field. We assume that no secondary flow is generated and that there is no variation, either in the duct cross-section or in the imposed magnetic field, with distance z along the duct. It is also postulated that any external circuit connected to the conducting walls of the duct is continuous and unvarying in the streamwise direction. (This condition may be relaxed if the conductivity is sufficiently small.) Thus all physical quantities except pressure are independent of z. Relative to the axes defined in figure 1, the equations describing such flows are:

$$j_x = \sigma\left(-\frac{\partial\phi}{\partial x} - v_z B_0\right), \quad j_y = \sigma\left(-\frac{\partial\phi}{\partial y}\right), \tag{2.1}$$

$$\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} = 0, \quad j_x = \frac{\partial H_z}{\partial y}, \quad j_y = -\frac{\partial H_z}{\partial x}, \tag{2.2}$$

$$0 = -\partial p/\partial z + j_x B_0 + \bar{\mu}\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right). \tag{2.3}$$

Here j_x, j_y are the components of the current, ϕ is the electric potential, H_z is the induced field and may also be regarded as a current stream function, B_0 is the flux density of the imposed magnetic field, v_z is the velocity, σ the conductivity, $\bar{\mu}$ the viscosity and $\partial p/\partial z$ the pressure gradient which is a constant. The equations can be re-written to give two coupled second-order partial differential equations in H_z and v_z, viz.

$$\bar{\mu}\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right) + B_0\frac{\partial H_z}{\partial y} - \frac{\partial p}{\partial z} = 0, \tag{2.4}$$

$$\frac{1}{\sigma}\left(\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2}\right) + B_0\frac{\partial v_z}{\partial y} = 0. \tag{2.5}$$

We take the lengths of the sides of the channel to be $2a$ and $2b$ (see figure 1) and suppose that the sides $y = \pm a$ (*BB*) are non-conducting, while the sides $x = \pm b$ (*AA*) are perfectly conducting. It follows that the boundary conditions are

$$v_z = 0, \quad \partial H_z/\partial x = 0 \quad \text{when} \quad y = \pm a, \tag{2.6}$$

$$v_z = 0, \quad \partial H_z/\partial x = 0 \quad \text{when} \quad x = \pm b. \tag{2.7}$$

Thus on the walls $y = \pm a$, H_z is independent of x and consequently we can modify (2.6) to

$$v_z = 0, \quad H_z = H_1 \quad \text{when} \quad y = a; \quad v_z = 0, \quad H_z = H_2 \quad \text{when} \quad y = -a, \tag{2.8}$$

where H_1, H_2 are constants. The net current I leaving and entering the walls AA per unit length of the duct is simply related to H_1, H_2:

$$I = \int_{-a}^{a} j_x dy|_{x=b} = H_1 - H_2. \tag{2.9}$$

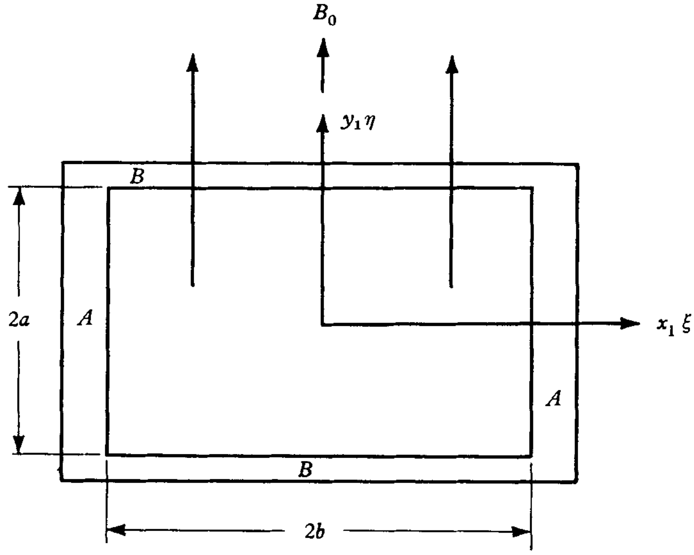

FIGURE 1. Cross-section of a rectangular duct with the magnetic field in the y-direction. The walls AA lie at $x = \pm b$ and BB at $y = \pm a$.

The governing equations and boundary conditions may now be reduced to a non-dimensional form by writing

$$\xi = x/a, \quad \eta = y/a, \quad M = aB_0(\sigma/\overline{\mu})^{\frac{1}{2}}, \tag{2.10}$$

$$-\frac{\partial p}{\partial z} + \frac{B_0}{2a}(H_1 - H_2) = P, \tag{2.11}$$

$$v_z = \frac{a^2 P}{\overline{\mu}} v(\xi, \eta), \quad H_z = \tfrac{1}{2}(H_1 + H_2) + \tfrac{1}{2}(H_1 - H_2)\eta + aP(\sigma/\overline{\mu})^{\frac{1}{2}} h(\xi, \eta). \tag{2.12}$$

The equations satisfied by v, h are

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} + M\frac{\partial h}{\partial \eta} = -1, \quad \frac{\partial^2 h}{\partial \xi^2} + \frac{\partial^2 h}{\partial \eta^2} + M\frac{\partial v}{\partial \eta} = 0, \tag{2.13}$$

subject to $\qquad v = 0, \quad h = 0 \quad \text{when} \quad \eta = \pm 1,$

and $\qquad v = 0, \quad \partial h/\partial \xi = 0 \quad \text{when} \quad \xi = \pm b/a = \pm c. \qquad (2.14)$

We are particularly interested in the properties of the solution when $M \gg 1$ and to find them we proceed by a heuristic argument, relying for justification on the consistency of the results. For large M the interior of the duct may be divided into five parts, as indicated in figure 2. These are:

(a) The core region consisting of the majority of the interior but excluding the neighbourhoods of the walls.

(b) The primary or Hartmann layers, of thickness $O(M^{-1})$, near the walls $\eta = \pm 1$ but excluding the regions distant $O(M^{-\frac{1}{2}})$ from the side walls $\xi = \pm c$. We use the word primary for these boundary layers to emphasize their control of the flow in the core (a) and to distinguish them from (c).

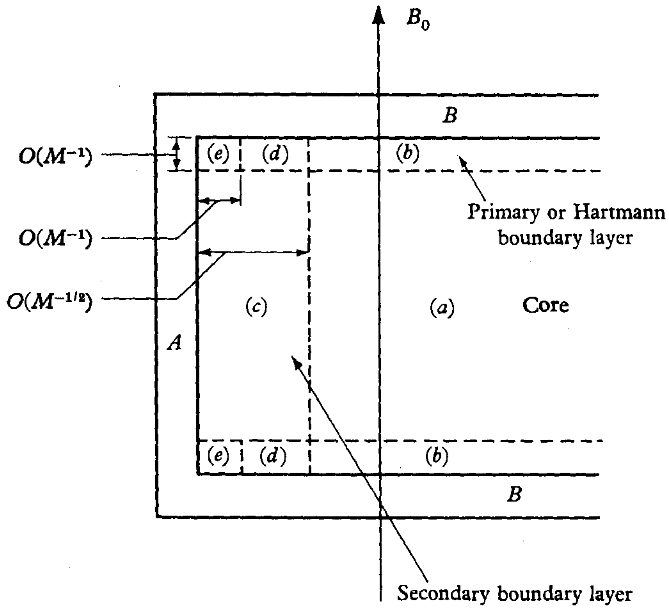

FIGURE 2. Cross-section of the duct showing the various regions of flow when $M \gg 1$ (not to scale).

(c) The secondary boundary layers, of thickness $O(M^{-\frac{1}{2}})$, near the walls $\xi = \pm c$, so called because they are determined from the core flow and the primary boundary layers but do not exert a decisive control on them in return.

(d) Those parts of the primary boundary layers at a distance $O(M^{-\frac{1}{2}})$ but $\gg M^{-1}$ from the side walls $\xi = \pm c$.

(e) Those parts of the interior of the duct within a distance $O(M^{-1})$ of the four corners.

Of these regions (e) is of the least importance and the most difficult to treat when M is large; we shall discuss it only by order of magnitude arguments. The other regions can, however, be discussed in detail as follows.

2.1. *Core flow* (a)

Since this region extends over almost the whole duct it follows that $\partial/\partial\xi$, $\partial/\partial\eta$ are $O(1)$. Further, from the boundary conditions and differential equations h must be odd and v even in η. Anticipating that v and h are of the same order of magnitude we then have from (2.13) that

$$h = h_0 \equiv -\eta/M, \quad v = v_c \equiv g(\xi)/M, \tag{2.15}$$

where $g(\xi)$ is a function of ξ to be determined, and terms of order M^{-2} have been neglected.

2.2. *The primary boundary layer (b) near* $\eta = 1$

The core solution (2.15) fails to satisfy the bundary conditions at the walls and in particular at $\eta = \pm 1$. Consequently there must be boundary layers near these walls to make the necessary adjustments in v and h and, since they are *ex hypothesis* thin, in them

$$\partial/\partial\eta \gg \partial/\partial\xi. \tag{2.16}$$

Writing

$$v = v_c + v_p, \quad h = h_c + h_p, \tag{2.17}$$

and concentrating attention on the boundary layer near $\eta = 1$, v_p and h_p satisfy

$$\frac{\partial^2 v_p}{\partial\eta^2} + M\frac{\partial h_p}{\partial\eta} = 0, \quad \frac{\partial^2 h_p}{\partial\eta^2} + M\frac{\partial v_p}{\partial\eta} = 0 \tag{2.18}$$

in virtue of (2.16) together with the boundary conditions

$$h_p = 1/M, \quad v_p = -g(\xi)/M \quad \text{at} \quad \eta = 1, \quad |\xi| < c, \tag{2.19a}$$

and

$$v_p \to 0, \quad h_p \to 0 \tag{2.19b}$$

on leaving the immediate neighbourhood of $\eta = 1$. A consistent solution of (2.18) satisfying (2.19) is only possible if

$$g(\xi) \equiv 1 \tag{2.20}$$

and then

$$v_p = -\frac{1}{M}e^{-M(1-\eta)}, \quad h_p = \frac{1}{M}e^{-M(1-\eta)}. \tag{2.21}$$

Thus the core velocity is determined by the condition for the existence of the primary boundary layer and, as anticipated in (2.15), is of the same order of magnitude as the induced magnetic field. Further the thickness of the primary boundary layer is $O(M^{-1})$, and the associated defect in velocity flux is

$$-\int_{-c}^{+c} d\xi \int_0^\infty v_p \frac{dY}{M} = 2c/M^2, \tag{2.22}$$

where $Y = (1-\eta)M$. The primary boundary layer near $\eta = -1$ may be treated by a parallel argument but we do not need to deal with it explicitly here since v is known to be even and h to be odd in η.

2.3. *The secondary boundary layer (c) near* $\xi = c$

The core solution is now fully known and does not satisfy the boundary conditions at $\xi = \pm c$. Consequently there must be boundary layers near these walls, to make the necessary adjustments in v and h and, since they are *ex hypothesis* thin, in them

$$\partial/\partial\xi \gg \partial/\partial\eta. \tag{2.23}$$

Writing

$$v = v_c + v_s, \quad h = h_c + h_s, \tag{2.24}$$

and concentrating attention on the boundary layer near $\xi = c$, v_s and h_s satisfy

$$\frac{\partial^2 v_s}{\partial\xi^2} + M\frac{\partial h_s}{\partial\eta} = 0, \quad \frac{\partial^2 h_s}{\partial\xi^2} + M\frac{\partial v_s}{\partial\eta} = 0 \tag{2.25}$$

in virtue of (2.23), together with the boundary conditions

$$\partial h_s/\partial \xi = 0, \quad v_s = -1/M \quad \text{at} \quad \xi = c, \quad |\eta| \leqslant 1, \tag{2.26a}$$

and

$$h_s \to 0, \; v_s \to 0 \tag{2.26b}$$

on leaving the immediate neighbourhood of $\xi = 1$. These obvious boundary conditions are, however, insufficient to solve the differential equations (2.25) completely. In addition we must know something about v_s, h_s at a station or stations of η. In the same way that region (b) provides the additional information to determine region (a), the regions (d) provide the extra boundary conditions needed here. We shall anticipate the discussion of regions (d) here and state the conditions

$$v_s + h_s \to 0 \quad \text{as} \quad \eta \to 1, \quad v_s - h_s \to 0 \quad \text{as} \quad \eta \to -1, \tag{2.27}$$

referring the reader to (2.52) below for their justification.

In order to solve (2.25) it is convenient to write

$$h_s = \alpha(\eta)/M \quad \text{at} \quad \xi = c, \tag{2.28}$$

which means that, in effect, we are specifying the current distribution on the walls AA. Further write

$$X = v_s + h_s, \tag{2.29}$$

and then, since v_s is even and h_s is odd in η, we have

$$v_s = \tfrac{1}{2}[x(\eta) + X(-\eta)], \quad h_s = \tfrac{1}{2}[X(\eta) - X(-\eta)]. \tag{2.30}$$

Now X satisfies

$$\frac{\partial^2 X}{\partial \xi^2} + M\frac{\partial X}{\partial \eta} = 0, \tag{2.31}$$

and the boundary conditions (2.26) and (2.28) become

$$X = \frac{1-\alpha(\eta)}{M}, \quad \frac{\partial}{\partial \xi}[X(\eta) - X(-\eta)] = 0 \quad \text{when} \quad \xi = c, \tag{2.32}$$

$X \to 0$ on leaving the vicinity of $\xi = c$, and $X = 0$ at $\eta = 1$. Leaving the condition on $\partial X/\partial \xi$ on one side for the moment the general solution for X is

$$X = -\frac{1}{M}\operatorname{erfc}\frac{(c-\xi)M^{\frac{1}{2}}}{2(1-\eta)^{\frac{1}{2}}} + \frac{c-\xi}{2(\pi M)^{\frac{1}{2}}}\int_\eta^1 \frac{\alpha(\eta_1)\,d\eta_1}{(\eta_1-\eta)^{\frac{3}{2}}}\exp\left\{-\frac{M(c-\xi)^2}{4(\eta_1-\eta)}\right\}. \tag{2.33}$$

To satisfy the condition on $\partial X/\partial \xi$, it follows on differentiating (2.33) with respect to ξ and setting $\xi = c$ that

$$\int_\eta^1 \frac{\alpha'(\eta_1)\,d\eta_1}{(\eta_1-\eta)^{\frac{1}{2}}} - \frac{\alpha(1)-1}{(1-\eta)^{\frac{1}{2}}} = \int_{-1}^{\eta} \frac{\alpha'(\eta_1)\,d\eta_1}{(\eta-\eta_1)^{\frac{1}{2}}} + \frac{\alpha(-1)+1}{(1-\eta)^{\frac{1}{2}}}. \tag{2.34}$$

Thus the problem has been reduced to finding the value of $\alpha(\eta)$ which leads to a constant electric potential on the walls AA.

The equation (2.34) may be cast into a recognizable form by writing

$$\alpha(\eta) = 1 - A(\eta)(1+\eta)^{\frac{1}{2}}. \tag{2.35}$$

Multiplying it by $(\zeta-\eta)^{-\frac{1}{2}}$ and integrating from -1 to ζ with respect to η:

$$A(\zeta) - \frac{1}{\pi}\int_{-1}^{+1}\frac{A(\eta)\,d\eta}{\eta-\zeta} = +\frac{2}{(\zeta+1)^{\frac{1}{2}}}. \tag{2.36}$$

This equation has a known solution (Rott & Cheng 1954) for a general right-hand side, which reduces in our case to

$$\alpha(\eta) = 1 - \frac{8(1-\eta^2)^{\frac{1}{2}}}{\pi(1+\eta)} \int_1^\infty \frac{s^2\, ds}{(s^4+\psi)\,(s^4-1)^{\frac{1}{2}}}, \tag{2.37}$$

where $\psi = (1-\eta)/(1+\eta)$. Thus α may be expressed in terms of a hypergeometric function. In particular when $\eta \approx 1$, ψ is small and

$$\alpha(\eta) = 1 - \frac{(1-\eta^2)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}} \frac{(-\frac{1}{4})!}{(\frac{1}{4})!} + O(1-\eta^2)^{\frac{3}{4}}. \tag{2.38}$$

$\alpha \to 0$ as $\eta \to 0$ and α is of course an odd function of η. Knowing $\alpha(\eta)$ we can calculate X, v_s and h_s from (2.33) and (2.32).

In an earlier paper by one of us (Hunt 1965) it was shown that, if the walls $\xi = \pm c$ are non-conducting and the walls $\eta = \pm 1$ perfect conductors, then the velocity in the secondary boundary layers is an order of magnitude greater than the core velocity and contains an infinite number of reversals of sign. If all four walls are perfect conductors, then the velocity also oscillates an infinite number of times in the secondary boundary layer, although in this case there are no reversals of sign and the velocity is of the same order of magnitude as in the core. It is of interest therefore to examine the nature of the boundary-layer flow in the present problem. At large distances (in terms of $(c-\xi)\, M^{\frac{1}{2}}$) from the wall $\xi = c$, the structure of the boundary layer in X is given by the behaviour of α near $\eta = 1$. From (2.38) $1-\alpha \sim (1-\eta)^{\frac{1}{4}}$ as $\eta \to 1-$ and hence, from the similarity solution of (2.31) satisfying

$$X = 0 \;\text{ at }\; \eta = 1, \quad X \to 0 \;\text{ as }\; \xi \to -\infty, \quad X \sim (1-\eta)^{\frac{1}{4}} \;\text{ at }\; \xi = c,$$

we find that when $(c-\xi)\, M^{\frac{1}{2}}$ is large

$$X \sim \frac{(1-\eta)^{\frac{9}{4}}}{M^2(c-\xi)^3} \exp\left\{ -\frac{M/(c-\xi)^2}{4(1-\eta)} \right\}, \tag{2.39}$$

so that the number of oscillations in v is at most finite. A graph of v as a function of $M^{\frac{1}{2}}(c-\xi)$ for $\eta = 0$ is given in figure 3 and shows that in fact v never changes sign. Lines of constant h are shown schematically in figure (5).

In order to calculate the overall velocity flux, we have to work out the flux deficits due to the boundary layers. Since h is an odd function of η, the flux deficit due to the secondary boundary layer is given by

$$-\int_{-1}^{+1} d\eta \int_0^\infty \frac{d\overline{\xi}}{M^{\frac{1}{2}}} v_s = -\int_{-1}^{+1} d\eta \int_0^\infty \frac{d\overline{\xi}}{M^{\frac{1}{2}}} X, \tag{2.40}$$

where $\overline{\xi} = (c-\xi)\, M^{\frac{1}{2}}$.

From (2.31)
$$-\int_0^\infty \frac{d\overline{\xi}}{M^{\frac{1}{2}}} X = \frac{1}{M(M\pi)^{\frac{1}{2}}} \int_\eta^1 \frac{(1-\alpha(\eta_1))\, d\eta_1}{(\eta_1-\eta)^{\frac{1}{2}}}, \tag{2.41}$$

so that the flux deficit is

$$\frac{2}{M(M\pi)^{\frac{1}{2}}} \int_{-1}^{+1} [1-\alpha(\eta)]\, [1+\eta]^{\frac{1}{2}}\, d\eta. \tag{2.42}$$

On substituting from (2.41) into (2.42) we obtain, after formal manipulation,

$$\frac{(-\frac{1}{4})!}{(+\frac{1}{4})!}\frac{2^{\frac{1}{2}}}{M^{\frac{3}{2}}} \tag{2.43}$$

for the flux deficit due to this boundary layer.

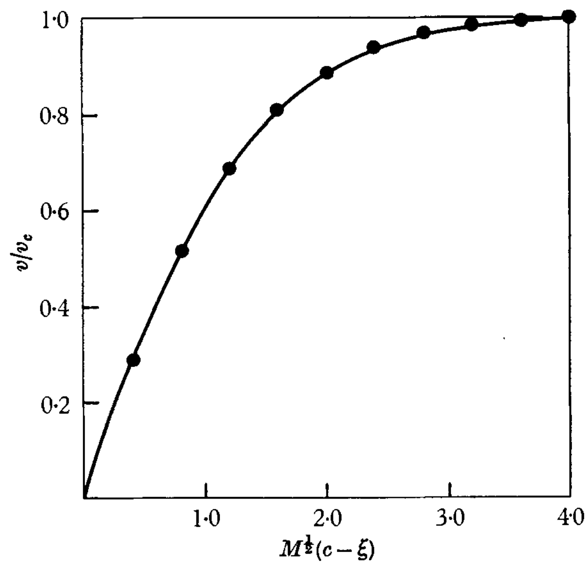

FIGURE 3. Graph of v/v_c against $M^{\frac{1}{2}}(c-\xi)$ at $\eta = 0$ in the boundary layer at $\xi = c$. The value of M is arbitrary, provided $M \gg 1$.

2.4. *Primary boundary layer (d) near the corner $\xi = c$, $\eta = 1$*

Taking

$$v = v_c + v_p + v_s, \quad h = h_c + h_p + h_s, \tag{2.44}$$

we satisfy the governing differential equations provided we can neglect terms $O(M^{-2})$. The boundary conditions on the walls $\xi = c$ and $\eta = 1$ are also satisfied, provided we can neglect exponentially small terms and provided we exclude the neighbourhood of the corner $\xi = c$, $\eta = 1$. Specifically the conditions are not satisfied when $\xi = c$ and $\eta = 1 - O(M^{-1})$ and when $\eta = 1$ and $\xi = c - O(M^{-\frac{1}{2}})$. It is the second of these with which we are concerned here and we shall briefly refer to the other in §2.5. If instead of (2.44) we write

$$v = v_c + v_p + v_s + v_{p_1}, \quad h = h_c + h_p + h_s + h_{p_1}, \tag{2.45}$$

where v_{p_1}, h_{p_1} satisfy the same differential equations as v, h, then, on the wall $\eta = 1$, we must have

$$v_{p_1} + v_s = 0, \quad h_s + h_{p_1} = 0. \tag{2.46}$$

Although the values of v_s, h_s on the wall $\eta = 1$ vary rapidly with ξ, the scale is much larger than the scale of the primary boundary layer, $O(M^{-\frac{1}{2}})$ compared with $O(M^{-1})$, and hence we are justified in assuming that v_{p_1}, h_{p_1} satisfy the condition $\partial/\partial\eta \gg \partial/\partial\xi$, whence

$$v_{p_1} = -v_s(\xi, 1)\, e^{-M(1-\eta)}, \quad h_{p_1} = -h_s(\xi, 1)\, e^{-M(1-\eta)}. \tag{2.47}$$

It follows that a consistent solution is only possible if

$$v_s(\xi, 1) + h_s(\xi, 1) = 0, \tag{2.48}$$

as assumed earlier in (2.27). The other condition in (2.27) follows from a parallel argument for the wall $\eta = -1$. It is noted that, if the wall $\eta = 1$ is a perfect conductor, $v = 0$, $\partial h/\partial \eta = 0$ there, and a parallel argument shows that the condition satisfied by v_s, h_s at $\eta = 1$ is then

$$v_s(\xi, 1) = 0. \tag{2.49}$$

2.5. *The corner* $\xi = 0, \eta = 1$

The assumptions leading to the primary boundary layer (*d*) and the secondary boundary layer (*c*) fail when both $1-\xi$ and $1-\eta$ are $O(M^{-1})$, i.e. in region (*e*). No simplification in the governing equations is possible therefore in this region. However its effect on the flux is small, being of the order of the maximum velocity multiplied by the area, i.e. $O(M^{-3})$, and consequently we have not attempted to elucidate its properties.

The leading terms in the asymptotic expansion for the flux of fluid through the tube when M is large may now be written down

$$\iint_{\text{duct}} v \, d\xi \, d\eta = \frac{4c}{M} - \frac{(-\frac{1}{4})!}{(+\frac{1}{4})!} \frac{2^{\frac{3}{2}}}{M^{\frac{3}{2}}} - \frac{4c}{M^2} + O(M^{-\frac{5}{2}}). \tag{2.50}$$

The term $O(M^{-\frac{5}{2}})$ arises partly from the neglect of $\partial^2 v/\partial \eta^2$, $\partial^2 h/\partial \eta^2$ in the secondary boundary layer and partly from the deficit due to the primary boundary layers (*d*). In principle it can be calculated using the methods of this paper but we have not done so.

3. Practical implications

In §2 it was shown that, whatever the value of I, the net current leaving and entering the duct, the problem of calculating the velocity and current distributions could be reduced to the solution of two differential equations with a single set of boundary conditions. It follows that the velocity *distribution* is always the same, though the *magnitude* of the velocity depends on the values of I, M and $\partial p/\partial z$. It follows from (2.12) that the distribution of current density does vary with the external circuits, but in a simple manner: j_y varies in magnitude but its distribution does not change: j_x has two constituents one of which, j_{x_1}, is constant throughout the duct, but varies with I, and is given by

$$j_{x_1} = \tfrac{1}{2}(H_1 - H_2)/a;$$

the other, j_{x_2}, whose *distribution* is always the same but whose *magnitude* varies, is given by

$$j_{x_2} = \frac{\sigma^{\frac{1}{2}} a P}{(\bar{\mu})^{\frac{1}{2}}} \frac{\partial H_z}{\partial y},$$

where $P = B_0 I/2a - \partial p/\partial z$.

In §2 we also found the velocity and current distributions and a relation between Q, $\partial p/\partial z$, I and M. In this section we use this information to examine the effect of electrically connecting the walls AA for the various practical applications of the duct.

The type of duct whose walls BB are non-conducting and walls AA are perfectly conducting has many applications. Shercliff (1965) has recorded how such a duct for which $b \gg a$ acts as a pump, flowmeter, generator or brake depending on the value of $-E_x/B_0V_m$, where E_x is the electric field in the core and V_m is the mean velocity; $V_m = V_c(1-1/M)$. If $b \sim a$ and the effects of the walls AA are considered, a new parameter has to be defined. In interpreting experiments or designing equipment the following five parameters are of most interest: $Q, \partial p/\partial z, B_0, I$, and $\Delta\phi$, the electric potential difference between the walls AA given by

$$\Delta\phi = \phi_{x=b} - \phi_{x=-b}. \tag{3.1}$$

Usually we are given three of these parameters and we wish to find the other two in terms of these, e.g. in designing an electromagnetic pump we would want to calculate I and $\Delta\phi$, given Q, $\partial p/\partial z$, B_0, and the fluid properties.

To find $\Delta\phi$ integrate equation (2.1) from $x = -b$ to $x = b$ and from $y = -a$ to $y = a$, which leads to

$$-\Delta\phi = \frac{b}{a\sigma}I + \frac{B_0 Q}{2a}. \tag{3.2}$$

This equation shows that to an external electric circuit the duct is equivalent to an e.m.f. $U_i = \frac{1}{2}B_0 Q/a$ in series with a resistance $R_i = b/a\sigma$. The replacement of a device in a d.c. electric circuit by an e.m.f. and a resistance is familiar to electrical engineers as Thevenin's theorem. By the same theorem any linear external circuit may be regarded as a resistance R_e and an e.m.f. U_e. Hence, in the general case $\Delta\phi$, $\partial p/\partial z$, Q, M and I may be calculated from the following relations:

$$\Delta\phi = -U_i - R_i I = U_e + R_e I,$$

or

$$\Delta\phi = -\tfrac{1}{2}B_0 Q/a - (b/\sigma a)\, I = U_e + R_e I, \tag{3.3}$$

and

$$Q = f(\tfrac{1}{2}B_0 I/a - \partial p/\partial z, M). \tag{3.4}$$

In §2 we calculated the function (3.4). See (2.50), which may be re-written as

$$Q = \left(\frac{B_0 I}{2a} - \frac{\partial p}{\partial z}\right)\frac{4a^3 b}{\overline{\mu} M}\left(1 - \frac{0{\cdot}956a}{bM^{\frac{1}{2}}} - \frac{1}{M} - \dots\right). \tag{3.5}$$

We now examine three special cases.

3.1. *Open-circuit case*

When the duct is on open circuit, $I = 0$. In this case the duct is a flowmeter. It follows from (3.5) that

$$Q = \frac{(-\partial p/\partial z)\, 4a^3 b}{\overline{\mu} M}\left(1 - \frac{0{\cdot}956a}{bM^{\frac{1}{2}}} - \frac{1}{M} - \frac{a}{b}O(M^{-\frac{3}{2}})\right), \tag{3.6}$$

and from (3.3) that

$$\Delta\phi = -\tfrac{1}{2}B_0 Q/a. \tag{3.7}$$

This fact has been noted many times before (see Shercliff 1962, p. 16).

3.2. *Short-circuit case*

In this case the walls AA are joined by a circuit of zero resistance ($R_e = U_e = 0$) and consequently $\Delta\phi = 0$. This device is an electromagnetic brake or generator on short-circuit.

Equation (3.3) becomes,

$$0 = \frac{2bI}{\sigma} + B_0 Q,$$

and hence (3.5) becomes

$$Q = \left(-\frac{B_0^2 Q\sigma}{4ab} - \frac{\partial p}{\partial z}\right)\frac{4a^3b}{\bar{\mu}M}\left(1 - \frac{0{\cdot}956a}{bM^{\frac{1}{2}}} - \frac{1}{M}\dots\right). \tag{3.8}$$

It follows that

$$Q = \frac{(-\partial p/\partial z)\,4a^3b}{\bar{\mu}M^2}\left[\frac{1 - \dfrac{0{\cdot}956a}{bM^{\frac{1}{2}}} - \dfrac{1}{M}\dots}{1 - \dfrac{0{\cdot}956a}{bM^{\frac{1}{2}}} - O(M^{-\frac{3}{2}})}\right],$$

and re-arranging

$$Q = \frac{(-\partial p/\partial z)\,4a^3b}{\bar{\mu}\,M^2}\left[1 - \frac{1}{M} - O(M^{-\frac{3}{2}})\right]. \tag{3.9}$$

Though the form of the $Q - \partial p/\partial z$ relation is different in this case from the open-circuit case, the velocity flux deficit due to the secondary boundary layers as a proportion of the flux through the core is the same, since the velocity distribution is unaffected by external connexions. There is no term of order $(M^{-\frac{1}{2}})$ in the bracket, as one might expect, because the core velocity V_c is given by

$$V_c = \frac{(-\partial p/\partial z)\,a^2}{\bar{\mu}M^2}\left(1 + \frac{0{\cdot}956a}{bM^{\frac{1}{2}}} + O(M^{-1})\right). \tag{3.10}$$

Thus the flow in the core is not the same as that for flow in a duct whose walls are all perfectly conducting. In that case

$$V_c = \frac{(-\partial p/\partial z)\,a^2}{\bar{\mu}M^2}.$$

This difference is explained by the fact that in the latter case $E_x = 0$ in the core, whereas in the former case, even though $\Delta\phi = 0$, $E_x \neq 0$ in the core because of the defect of $\mathbf{v}\times\mathbf{B}$ in the secondary boundary layers.

3·3. *Purely electrically driven case*

When the pressure gradient is zero and the flow is electrically driven by the potential difference across the walls AA, the device is a limiting form of MHD pump. Since $\partial p/\partial z = 0$, equation (3.5) becomes

$$Q = \frac{\frac{1}{2}B_0 I a^2 b}{\bar{\mu}M}\left(1 - \frac{a}{b}\frac{0{\cdot}956}{M^{\frac{1}{2}}} - \frac{1}{M} + \dots\right). \tag{3.11}$$

Combining this result with (3.2) leads to

$$Q = -\frac{2a\Delta\phi}{B_0}\left[1 - \frac{1}{M} - O\left(\frac{a}{bM^{\frac{3}{2}}}\right)\right]. \tag{3.12}$$

In the two previous cases either I or $\Delta\phi$ is zero, whereas in this case we can obtain a useful relation between I and $\Delta\phi$ by dividing (3.12) by (3.11),

$$\frac{\Delta\phi}{I} = -\frac{Mb}{a\sigma}\left[1 - \frac{0{\cdot}956a}{bM^{\frac{1}{2}}} - O\left(\frac{a}{bM^{\frac{3}{2}}}\right)\right]. \tag{3.13}$$

It is interesting to note that in the core $j_x = 0$ since $\partial p/\partial z = 0$. All the current is in the primary and secondary boundary layers (see figure 5 (*b*)).

From an examination of these three special cases it follows that the parameter which describes the particular application of a duct for which $b \sim a$ is $-2a\Delta\phi/B_0 Q$ as opposed to $-E_x/B_0 V_m$ in a duct for which $b \gg a$.

4. Discussion

In §2, we analysed the flow through ducts whose walls AA were perfectly conducting and walls BB were non-conducting. The flow was assumed to be laminar, incompressible, uniformly conducting, and fully developed; also the Hartmann number (M) was assumed to be large. In §3, we showed how the results of the analysis could be used for ducts with various electrical connexions between the walls AA. In this section we compare the previous results with those obtained for other types of duct and we show how some of the effects of the conductivity of the walls on the flow may be interpreted physically.

For a duct whose walls AA are perfectly conducting and walls BB are non-conducting, we have been able to analyse the flow only for very large Hartmann number. For our purposes this is no great disadvantage since a solution at high M is sufficient to illustrate the essential physical features of the flow and also it is quite usual in liquid metal experiments to have M greater than 100. In this discussion we shall concentrate on flows with $M \gg 1$.

At high M, provided the conductivity of each of the walls BB is constant along its length, the value of the conductivity of each of the walls BB does not affect the *form* of the velocity profile away from the walls AA. The velocity is constant except in the narrow primary or Hartmann boundary layers on $y = \pm a$. However, the *magnitude* of the velocity in the core depends on the conductivity of the walls BB as well as on the pressure gradient and any external electrical circuit connected between the walls AA.

Though boundary layers are also found on the walls AA as $M \to \infty$, their form changes considerably with the conductivity of the duct walls. It is these secondary boundary layers which have been little understood hitherto. The three main characteristics of these boundary layers are the velocity profile, the current distribution and the velocity flux deficit, as defined in (2.40). It does not seem possible to provide convincing explanations for the shape of the velocity profiles, *a priori*, but it is possible to provide a physical explanation for the current distribution and velocity flux deficit and thence to explain the shape of the velocity profile.

Let us now compare the flows in two types of rectangular duct:

Case (i) Perfectly conducting walls all around.

(ii) Walls AA perfectly conducting and walls BB non-conducting.

In case (i) the velocity profile in the secondary boundary layers has the form of a damped sine wave and the velocity flux is given by

$$Q = \frac{(-\partial p/\partial z)\, 4a^3 b}{\bar{\mu} M^2}\left[1 - \frac{1}{M} - \frac{2{\cdot}43a}{bM^{\frac{1}{2}}} + \dots\right],$$

so that the flux deficit is $O(M^{-\frac{1}{2}})$ times the flux in the core (see Hunt 1965); in case (ii) the velocity in the secondary boundary layers monotonically decreases to zero at the wall, and the flux deficit is $O(M^{-\frac{1}{2}})$ times the core velocity (see § 2). These values of flux deficit were obtained by mathematical rather than physical arguments. To show that the difference in flux deficit between cases (i) and (ii) is explicable physically, we now give arguments by which the orders of magnitude of the flux deficits are estimated.

4.1. *All the walls are perfectly conducting—case (i)*

We consider the secondary boundary layer on the wall A at $x = -b$ (see figure 4). Let

$$j_x = j_c + j_p + j_s, \tag{4.1}$$

$$v_z = v_c + v_p + v_s, \tag{4.2}$$

where, as before, the suffices c, p and s refer to the core, primary and secondary boundary layers, respectively.

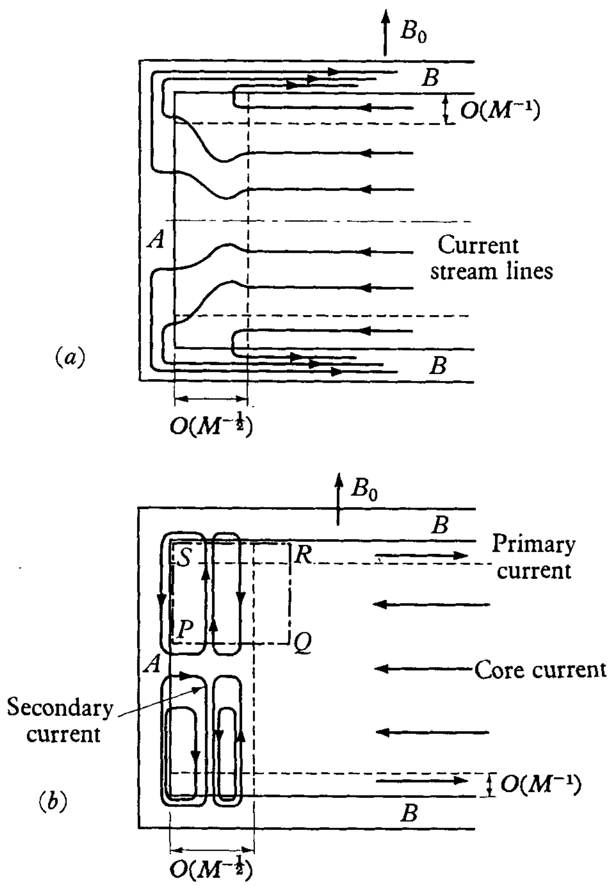

FIGURE 4. Cross-section of the duct when all the walls are perfectly conducting. ($M \gg 1$, not to scale). (*a*) Shows the actual current streamlines. (*b*) Shows the core, primary and secondary current streamlines.

Making the usual assumption that in the boundary layer on walls AA

$$\partial/\partial x \gg \partial/\partial y, \quad \text{for} \quad (a-y) \gg aM^{-1},$$

equation (2.3) leads to

$$0 = -\frac{\partial p}{\partial z} + (j_c + j_s)\, B_0 + \bar{\mu}\,\frac{\partial^2}{\partial x^2}(v_c + v_s). \tag{4.3}$$

Since $j_c B_0 = \partial p/\partial z$ and $\partial v_c/\partial x = 0$, (4.3) becomes

$$0 = j_s B_0 + \bar{\mu}\,\frac{\partial^2 v_s}{\partial x^2}. \tag{4.4}$$

Let the thickness of the boundary layer be δ, then

$$\frac{\partial^2 v_s}{\partial x^2} = -O\left(\frac{v_c}{\delta^2}\right) \quad \text{and} \quad j_s = O\left(\frac{\bar{\mu} v_c}{B_0 \delta^2}\right). \tag{4.5}$$

Now since the current in the core enters the wall A, $E_x|_{x=-b} = O(j_c/\sigma)$. Since $E_x = 0$ at $y = \pm a$, $\partial E_x/\partial y = O(-j_c/a\sigma)$ for $y > 0$. But

$$\partial E_y/\partial x = O(j_y/\sigma\delta) \quad \text{and} \quad \partial E_x/\partial y = \partial E_y/\partial x,$$

and hence $j_c/a = -O(j_y/\delta)$. But $\partial j_y/\partial y = -\partial j_s/\partial x$ since $\partial j_c/\partial x = 0$. Hence

$$j_s/\delta = O(j_y/a) = O\,(-j_c\delta/a^2)$$

and

$$j_s = -O(\delta^2 j_c/a^2). \tag{4.6}$$

From equations (4.5) and (4.6) we have,

$$\bar{\mu} v_c/\delta^2 = O(-\delta^2 B_0 j_c/a^2). \tag{4.7}$$

Since $E_x = 0$ in the core,

$$B_0 v_c = -j_c/\sigma. \tag{4.8}$$

Combining (4.7) and (4.8) we deduce that

$$\delta = O(aM^{-\frac{1}{2}}). \tag{4.9}$$

Using (4.9) we now have an expression for j_s:

$$j_s = O(\bar{\mu} M v_c/a^2 B_0). \tag{4.10}$$

Now consider $\oint E.d\mathbf{l}$ taken round the path $PQRS$ on figure 4(*b*). Since $E_y = 0$ in the core and along PS, and $E_x = 0$ along RS,

$$\oint_{PQRS} \mathbf{E}.d\mathbf{l} = \int_P^Q E_x d_x.$$

But in steady flow $\oint \mathbf{E}.d\mathbf{l} = 0$. Therefore $\int_P^Q E_x d_x = 0$, and hence

$$\int_0^\delta \frac{(j_c + j_s)}{\sigma}\, dx + B_0 \int_0^\delta (v_c + v_s)\, dx = 0.$$

But from (4.8), $j_c/\sigma + B_0 v_c = 0$. Then, using (4.10) and assuming that j_s is mainly positive so that $\int_0^\delta j_s dx = O(\delta j_s)$,

$$\int_0^\delta v_s dx = -O\left(\frac{\bar{\mu} M v_c \delta}{\sigma a^2 B_0^2}\right),$$

and
$$\int_{-a}^{a}\int_0^\delta v_s dx dy = -O(v_c a^2/M^{\frac{3}{2}}). \tag{4.11}$$

Hence the ratio of the flux deficit due to the secondary boundary layer to the flux in the core is $O[M^{-\frac{3}{2}}]$, in agreement with the exact analysis.

4.2. *Walls AA perfectly conducting and walls BB non-conducting—case (ii)*

The notation is the same as for case (i). We can then proceed to equation (4.5) as before. In this case the walls BB are non-conducting and consequently

$$\int_{-a}^{+a} j_x dy = I$$

is a constant for all values of x (see figure 5).

Therefore all the secondary current j_s leaving the wall A will have to return at the corners via the region (d) on wall B. If $j_s = j_s'$ in region (d), then, for condition (4.12) to be satisfied,

$$j_s' a/M = O(-a j_s),$$

and hence
$$j_s' = O(-M j_s). \tag{4.12}$$

We can deduce the value of j_s' from the equation of motion in region (d). If $v_s = v_s'$ in this region, then

$$0 = -\partial p/\partial z + (j_c + j_p + j_s') B_0 + \bar{\mu}\frac{\partial^2}{\partial y^2}(v_c + v_p + v_s').$$

The boundary conditions on v_s are (i) $v_s' = v_s = -0(v_c)$ in the main part of the secondary boundary, layer, i.e. region (c), and (ii) $v_s = 0$ on $y = \pm a$. Since the thickness of this layer is $O(M^{-1})$,

$$j_s' = -O\left(\frac{\bar{\mu}}{B_0}\frac{v_c}{(a/M)^2}\right). \tag{4.13}$$

Hence, from (4.12),
$$j_s = +O\left(\frac{\bar{\mu}}{B_0} M v_c/a^2\right). \tag{4.14}$$

But, from (4.5),
$$j_s = O\left(\frac{\bar{\mu}}{B_0} v_c/\delta^2\right),$$

and therefore
$$\delta = O(aM^{-\frac{1}{2}}).$$

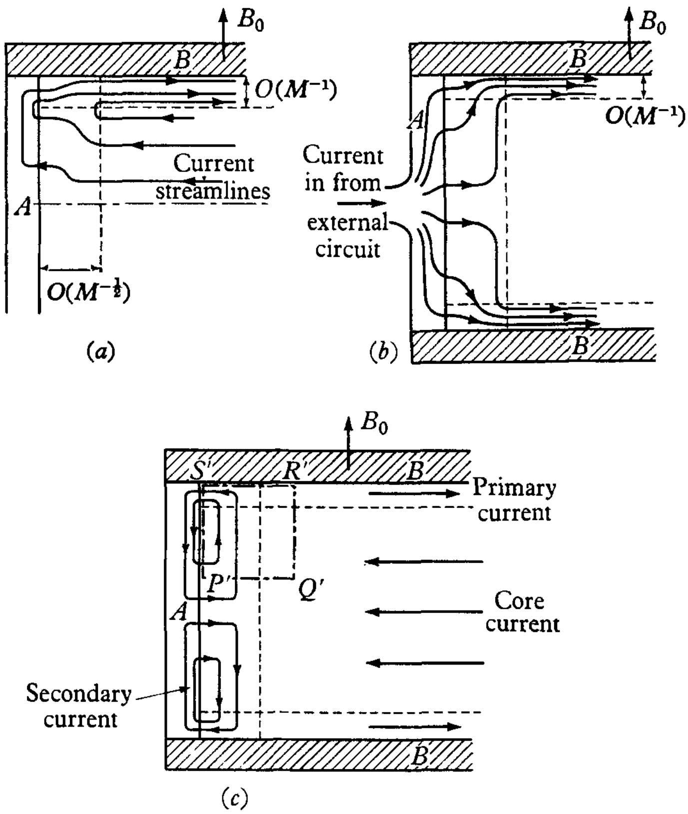

FIGURE 5. Cross-section of the duct when the walls AA are perfectly conducting and the walls BB are non-conducting ($M \gg 1$, not to scale). (*a*) Shows the current streamlines when the duct is on open circuit. (*b*) Shows the current streamlines when the flow is driven by a potential difference between the walls AA. (The current in the walls AA is shown schematically.) (*c*) Shows the core, primary and secondary currents when the duct is on open circuit.

Now consider $\oint \mathbf{E}.d\mathbf{l}$ round $P'Q'R'S'$, where P' S' are on the wall A, R' is on the wall B outside the secondary boundary layer and Q' is in the core (figure 5(*b*)). Since $E_y = 0$ in the core and along the wall A,

$$\int_{S'}^{R'} E_x d_x = \int_{P'}^{Q'} E_x d_x, \quad \text{since} \quad \oint \mathbf{E}.d\mathbf{l} = 0.$$

Now
$$\int_{S'}^{R'} E_x d_x = \int_0^\delta \frac{j_s'}{\sigma} dx,$$

and
$$\int_{P'}^{Q'} E_x d_x = \int_0^\delta \frac{j_s}{\sigma} dx + B_0 \int_0^\delta v_s d_x.\dagger$$

Making the same assumption as for case (i), that $\int_0^\delta j_s' dx = O(\delta j_s')$ we have

$$\int_0^\delta \int_{-a}^{a} v_s dx dy = -O(v_c a^2/M^{\frac{1}{2}}).$$

† Thus $\int_0^\delta E_x dx \neq 0$ in this case, whereas $\int_0^\delta E_x dx = 0$ in case (i). The consequences of this were discussed in § 3.

Hence the flux deficit due to the secondary boundary layer is $O(M^{-\frac{1}{2}})$ times the flux in the core.

We see from these order-of-magnitude arguments that the form of the boundary layer on the walls AA is best explained in terms of the secondary currents induced in these layers, much in the same way as the Hartmann layer may be explained by the decrease in current in the boundary layer relative to that in the core. In both these types of boundary layer the current is less than its value in the core because of the reduced $\mathbf{v} \times \mathbf{B}$ induced electric field and consequently the electromagnetic $\mathbf{j} \times \mathbf{B}$ force can decrease in the layer relative to its value in the core to the same extent that the viscous stress increases. If this were not so, the boundary layers would grow or diminish. A comparison of cases (i) and (ii) shows that the value of the secondary currents relative to the core currents can differ by an order of magnitude.

In case (i), $j_s = -O(j_c M^{-1})$.
In case (ii), open circuit, $j_s = -O(j_c)$.
In case (ii), closed circuit, $j_s = -O(j_c M^{-1})$.

Yet expressed as a fraction of v_c, the values of j_s are of the same order in both cases. This is necessary for the balancing of viscous and electromagnetic forces in the boundary layer. We make this observation because the approximation made by Kerrebrock (1961) and others that the current is constant through the secondary boundary layer, i.e. $j_s = 0$, even though j_s is a very small fraction of j_c, will lead to results which may over-estimate the rate of growth of boundary layers on the walls AA.

The order-of-magnitude arguments also show how the conductivity of the walls affects the secondary currents which in turn affect the velocity distribution in the duct. In case (ii) when the walls BB are non-conducting, the secondary currents circulate in the duct and walls AA; they leave the walls AA in region (*c*) and return through the Hartmann layer on walls BB, region (*d*). This is similar to the way in which the core currents return through the Hartmann boundary layer when the walls BB are non-conducting (figure 5(*c*)). In case (i), when the walls BB are perfectly conducting, the secondary currents mainly return to the walls AA via the walls BB. Owing to the oscillatory nature of the boundary layer in this case, some currents circulate solely in the duct and walls BB (see figure 4(*b*)). Thus in case (ii), because the secondary current has to return through a narrow layer of thickness $O(M^{-1})$ on walls BB, the resistance of the current path is *higher* than in case (i); hence the secondary current, relative to the core velocity, is *lower*. This means that the viscous stresses in the secondary boundary layer are less in case (ii) than case (i). Since v_s has to rise from $-v_c$ to 0 as $M^{\frac{1}{2}}(x+b)$ increases from 0 to ∞, then the *lower* $\partial^2 v_s/\partial x^2$ and $\partial v_s/\partial x$ are, the *greater* will be $\int_0^\infty v_s dx$, i.e. the flux deficit. As we have already noted, in case (i), the flux deficit in the secondary boundary layer relative to the flux in the core is $O(M^{-1})$ times as small as the flux deficit in case (ii). We see now that the explanation lies in the influence on the secondary currents of the conductivity of the walls BB in the corner regions.

This may be illustrated further by comparing case (i) and case (ii) when the walls AA are short-circuited. The $Q-\partial p/\partial z$ relation in these two cases are both of the form

$$Q = \frac{(-\partial p/\partial z)\,4a^3b}{\bar{\mu}M^2}\left(1-\frac{1}{M}-\frac{a}{b}O(M^{-\frac{3}{2}})\ldots\right)$$

even though the velocity profiles are very different (see §3). If we were to alter the duct of case (ii) and make the walls BB conducting for a distance $O(aM^{-\frac{1}{2}})$ from the corners, then we would not alter the $Q-\partial p/\partial z$ relation but we would change the velocity profile, the core velocity from

$$\frac{(-\partial p/\partial z)\,a^2}{\bar{\mu}M^2}\left[1+\frac{0{\cdot}956a}{bM^{\frac{1}{2}}}\right] \quad \text{to} \quad \frac{(-\partial p/\partial z)\,a^2}{\bar{\mu}M^2},$$

and the ratio of the flux deficit due to the secondary boundary layers to the flux in the core from $O(M^{-\frac{1}{2}})$ to $O(M^{-\frac{3}{2}})$. The reason is that the extra pieces of conductor would increase the secondary currents and hence reduce the flux deficit and potential difference across the secondary boundary layer. The value of the core velocity would be reduced, but the form of the core and primary layer flow would of course be unaltered.

We have stated already that we can give no good reasons, *a priori*, for the shape of the velocity profile in the secondary boundary layers; all we can do is to deduce the shape from the flux deficit. In cases (ii) and (iii) the flux deficit is $O(M^{-\frac{1}{2}})$. Since the thickness of the secondary boundary layers is $O(aM^{-\frac{1}{2}})$, there is no reason to expect that the velocity does not uniformly decrease from its value in the core to zero at the walls. In case (i) the flux deficit is $O(M^{-\frac{3}{2}})$ and the boundary layer thickness is still $O(M^{-\frac{1}{2}})$. This explains why the velocity in the boundary layer must be greater than its value in the core over part of the boundary layer. The velocity profiles for case (i) are given by Hunt (1965).

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REFERENCES

Chang, C. C. & Lundgren, T. S. 1961 *Z. angew. Math. Phys.* **12**, 100.
Grinberg, G. A. 1961 *Appl. Math. and Mech. (PMM)*, **25**, 1536.
Grinberg, G. A. 1962 *Appl. Math. and Mech. (PMM)*, **26**, 106.
Hale, F. J. & Kerrebrock, J. L. 1964 *AIAA J.* **2**, 461.
Hunt, J. C. R. 1965 *J. Fluid Mech.* **21**, 577.
Kerrebrock, J. L. 1961 *J. Aero/Space Sci.* **28**, 631.
Neuringer, J. L. & Migotsky, E. 1963 *Phys. Fluids*, **6**, 1164.
Resler, E. J. & Sears, W. R. 1958 *J. Aero. Sci.* **25**, 235.
Rott, N. & Cheng, H. K. 1954 *J. Rat. Mech. and Anal.* **3**, 357.
Shercliff, J. A. 1962 *The Theory of Electromagnetic Flow Measurement.* Cambridge University Press.
Shercliff, J. A. 1965 *A Textbook of Magnetohydrodynamics.* Oxford: Pergamon Press.
Sutton, G. W. & Carlson, A. W. 1961 *J. Fluid Mech.* **11**, 121.
Uflyand, Y. S. 1961 *Sov. Phys. Tech. Phys.* **5**, 1194.